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By

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Linear functional-differential equations with constant coefficients.

Jack K. Hale

I. Introduction. Until recently, most of the results concerning differential-difference equations have been obtained by treating the dependent variable as a point in Euclidean space and employing arguments which are standard in the theory of ordinary differential equations. To the author's knowledge, Krasovskii [9] was the first to exploit the idea that the proper setting for these problems is in a function space. In doing so, the arguments used for ordinary differential equations become more natural for differential-difference equations. The present paper is an attempt to obtain some analogies between linear differential-difference equations with constant coefficients and ordinary linear differential equations with constant coefficients.

More specifically, we discuss in detail the eigenspaces of the linear equation and then make use of the adjoint equation to introduce new coordinates in the function space which exhibit in a natural manner the behavior of the solutionson an eigenspace and the behavior of the solutions on a complementary space. In this manner, it is shown in section IV how many of the usual perturbation theorems in ordinary differential equations can be easily extended to differential-difference equations. The basic idea for discussing the problems in this manner is contained in the papers of Shimanov (see the bibliography) and the present paper originated from an attempt to understand the geometric significance of Shimanov's results. This approach should lead to a better understanding of much of the geometric theory of differential-difference equations. The

author is indebted to John Steulpnagel and Arnold Stokes for many fruitful discussions.

The following notation will be used throughout this paper. R^n is the linear space of n-vectors and for $x \in R^n$, $|x|$ is any vector norm. For any given numbers α, β, $\alpha \leq \beta$, $C([\alpha, \beta], R^n)$ will denote the space of continuous functions mapping the interval $[\alpha, \beta]$ into R^n and for $\varphi \in C([\alpha, \beta], R^n)$, $\|\varphi\| = \sup_{\alpha \leq \theta \leq \beta} |\varphi(\theta)|$. For any $r \geq 0$, any continuous function $x(u)$ defined on $-r \leq u \leq A$, $A > 0$, and any fixed t, $0 \leq t \leq A$, we shall let the symbol x_t denote the function $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$; that is, $x_t \in C([\alpha, \beta], R^n)$ and is that "segment" of the function $x(u)$ defined by letting u range in the interval $t - r \leq u \leq t$.

Let $X(\varphi, t) \in R^n$ be a function defined for all $\varphi \in C([-r,0], R^n)$, $\|\varphi\| \leq H$, $H > 0$, $t \in [0,\infty)$. Let $\dot{x}(t)$ denote the right hand derivative of a function $x(u)$ at $u = t$, and consider the <u>functional-differential equation</u>

$$\dot{x}(t) = X(x_t, t). \tag{1.1}$$

<u>Definition I.1</u>. Let t_o be any given number ≥ 0 and let $\varphi \in C([-r, 0], R^n)$, $\|\varphi\| \leq H$, be any given function. A function $x_t(t_o, \varphi)$ is said to be <u>a solution of (1.1) with initial function φ at t_o</u> if there is a number $A > 0$ such that

i) for each t, $t_o \leq t \leq t_o + A$, $x_t(t_o, \varphi)$ is defined, belongs to $C([-r, 0], R^n)$ and $\|x_t(t_o, \varphi)\| \leq H$;

ii) $x_{t_o}(t_o, \varphi) = \varphi$;

iii) $x(t_o, \varphi)$ satisfies (1.1) for $t_o \leq t \leq t_o + A$.

If t_o is equal to zero, we shall abbreviate $x(t_o, \varphi)$ by $x(\varphi)$. If $X(\varphi, t)$ is continuous in φ, t and Lipschitzian in φ, it is easy to prove that (1.1) always has a solution and for each φ there is only one solution. Furthermore, it is also easy to prove that $x(t_o, \varphi)$ depends continuously on φ.

By a <u>linear functional-differential equation with constant coefficients</u>, we mean a system (1.1) where $X(\varphi, t) = f(\varphi)$ is homogeneous and additive in φ. It is well known [15, p. 110] that $f(\varphi)$ continuous on $C([-r, 0], R^n)$ implies there is a matrix $\eta(\theta)$ whose elements are of bounded variation such that

$$f(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta),$$

for all $\varphi \in C([-r, 0], R^n)$, where the integral is in the sense of Stieltjes. This observation makes it obvious that the concept of linear functional-differential equation with constant coefficients includes all linear differential-difference equations with constant coefficients of the form

$$\dot{x}(t) = \sum_{k=1}^{p} A_k x(t - \tau_k), \quad \tau_k \geq 0.$$

II. Basic properties of linear systems with constant coefficients. A linear functional-differential equation with constant coefficients is any equation of the form

$$(2.1) \qquad \dot{x}(t) = f(x_t)$$

where f is a continuous linear function mapping $C([-r, 0], R^n)$ into R^n. For any such function $f(\varphi)$, it is well known (see [15, p. 110]) that there exists an $n \times n$ matrix $\eta(\theta)$, $-r \leq \theta \leq 0$, whose elements have bounded variation such that

$$(2.2) \qquad f(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta).$$

If φ is any given function in $C([-r, o], R^n)$ and $x(\varphi)$ is the solution of (2.1) with the initial function φ at zero, we define the operator $\mathcal{J}(t)$ mapping $C([-r, 0], R^n)$ into $C([-r, 0], R^n)$ by the relation

$$(2.3) \qquad x_t(\varphi) = \mathcal{J}(t)\varphi,$$

where, for each fixed $t \geq 0$, $x_t(\varphi)$ is the function in $C([-r, 0], R^n)$ determined by the relation $x_t(\varphi)(\theta) = x(\varphi)(t + \theta)$, $-r \leq \theta \leq 0$.

Lemma II.1. The operator $\mathcal{J}(t)$, $t \geq 0$, defined on $C([-r, 0], R^n)$ by (2.3) satisfies the following properties.

i) $\mathcal{T}(t)$ is a bounded linear operator for each $t \geq 0$;

ii) $\mathcal{T}(t)$ is strongly continuous on $[0, \infty)$; that is $\mathcal{T}(0) = I$ and

$$\lim_{\tau \to t} \| \mathcal{T}(\tau)\varphi - \mathcal{T}(t)\varphi\| = 0,$$

for all $t \geq 0$, $\varphi \in C([-r, 0], R^n)$;

iii) The family of transformations $\{\mathcal{T}(t), t \geq 0\}$ is a <u>semigroup</u>, that is,

$$\mathcal{T}(t + \tau) = \mathcal{T}(t)\mathcal{T}(\tau), \quad \text{for all } t \geq 0, \tau \geq 0;$$

iv) $\mathcal{T}(t)$ is <u>completely continuous</u> (compact) for $t \geq r$; that is, $\mathcal{T}(t)$, $t \geq r$ maps closed bounded sets into compact sets.

<u>Proof</u>: i) It is obvious that $\mathcal{T}(t)$ is linear. Since $f(\varphi)$ is continuous and linear, it follows that there is a constant L such that $|f(\varphi)| \leq L\|\varphi\|$ for all φ. From the definition of $\mathcal{T}(t)$, we have, for any fixed t

$$\begin{aligned} &\mathcal{T}(t)\varphi(\theta) = \varphi(t + \theta), \quad t + \theta \leq 0, \\ (2.4) \qquad &\mathcal{T}(t)\varphi(\theta) = \varphi(0) + \int_0^{t+\theta} f(\mathcal{T}(\tau)\varphi)d\tau, \quad t + \theta > 0, \ -r \leq \theta \leq 0. \end{aligned}$$

Since $|f(\varphi)| \leq L\|\varphi\|$, it follows that

$$\| \mathcal{T}(t)\varphi\| \leq e^{Lt}\|\varphi\|, \quad t \geq 0, \quad \varphi \in C([-r, 0], R^n),$$

and $\mathcal{T}(t)$ is bounded.

ii) From i), it follows that $\mathcal{T}(t)$ is continuous for all $t \geq 0$ and, from the definition of $\mathcal{T}(t)$, $\mathcal{T}(0) = I$. This proves ii).

iii) This is immediate from the definition.

iv) To prove iv), we observe that if $S = \{\varphi \in C([-r, 0], R^n) \mid \|\varphi\| \leq R\}$ then

$$\mathcal{T}(t)S \subset S_1 = \{\psi \in C([-r, 0], R^n) \mid \dot\psi \in C([-r,0],R^n),\ \|\psi\| \leq e^{Lt}R,\ \|\dot\psi\| \leq Le^{Lt}R\}$$

for $t \geq r$. Since S_1 is compact and $\mathcal{T}(t)$ is continuous, the result follows. This completes the proof of Lemma II.1.

For any semigroup of transformations $\mathcal{T}(t)$, $t \geq 0$, of a Banach space $\mathcal{B}$ into itself, the infinitesimal generator $\mathcal{Q}$ of $\mathcal{T}(t)$ is defined by the relation

$$\mathcal{Q}\varphi = \lim_{t \to 0^+} \frac{1}{t}[\mathcal{T}(t)\varphi - \varphi]$$

for every value of φ for which this limit exists. The limit of course signifies convergence in the norm of $\mathcal{B}$.

For any operator $\mathcal{T}$ of a Banach space $\mathcal{B}$ into itself, the <u>resolvent</u> set $\rho(\mathcal{T})$ of $\mathcal{T}$ is the set of values λ in the complex plane for which the operator $\lambda I - \mathcal{T}$ has an inverse which is defined for all φ in $\mathcal{B}$. The complement of $\rho(\mathcal{T})$ in the complex plane is called the <u>spectrum</u> of $\mathcal{T}$ and is denoted by $\sigma(\mathcal{T})$. The spectrum $\sigma(\mathcal{T})$ of an operator consists of three different types of points, namely the <u>residual spectrum</u> $R\sigma(\mathcal{T})$, the <u>continuous spectrum</u> $C\sigma(\mathcal{T})$, and the <u>point spectrum</u> $P\sigma(\mathcal{T})$. The residual spectrum consists of those values of λ in $\sigma(\mathcal{T})$ for which

$\lambda I - \mathcal{T}$ exists but the domain $\mathcal{D}(\lambda I - \mathcal{T})^{-1}$ of $(\lambda I - \mathcal{T})^{-1}$ is not dense in $\mathcal{B}$. The continuous spectrum consists of those λ in $\sigma(\mathcal{T})$ for which $\mathcal{D}(\lambda I - \mathcal{T})^{-1}$ is dense in $\mathcal{B}$ and the point spectrum consists of those values of λ for $\lambda I - \mathcal{T}$ does not have an inverse. The points λ in $P\sigma(\mathcal{T})$ are sometimes called the <u>eigenvalues</u> of $\mathcal{T}$ and any non-zero φ such that $(\lambda I - \mathcal{T})\varphi = 0$ is called an <u>eigenvector</u>.

One of our first objectives is to try to determine the nature of $\sigma(\mathcal{T}(t))$ and $\sigma(\mathcal{A})$ for the family of operators which arise in our particular problem and to analyze in what sense the operator $\mathcal{T}(t)$ is approximated by the operator $e^{\mathcal{A}t}$ provided this latter object makes sense. For the simple case in which system (1) is an ordinary differential equation; that is, $f(\varphi) = A\varphi(0)$ for some constant matrix A, the operator $\mathcal{T}(t)$ is e^{At} and the infinitesimal generator $\mathcal{A}$ of $\mathcal{T}(t)$ is equal to the matrix A. The following results show that analogous results are valid for the more general system (2.1).

The following lemma is a restatement of Theorem 10.3.1 and 10.3.3 of Hille and Phillips [8] for our particular case.

<u>Lemma II.2</u>. If $\mathcal{T}(t)$, $t \geq 0$, is a strongly continuous semigroup of operators mapping $C([-r, 0], R^n)$ into $C([-r, 0], R^n)$, then the domain $\mathcal{D}(\mathcal{A})$ of the infinitesimal generator $\mathcal{A}$ of $\mathcal{T}(t)$ is dense in $C([-r, 0], R^n)$ and the range $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is in $C([-r, 0], R^n)$. For all φ in $\mathcal{D}(\mathcal{A})$,

$$\frac{d}{dt}\mathcal{T}(t)\varphi = \mathcal{T}(t)\mathcal{A}\varphi = \mathcal{A}\mathcal{T}(t)\varphi. \tag{2.5}$$

We now derive a specific formula for the infinitesimal generator $\mathcal{A}$ in terms of the system (2.1). Since $x(\varphi)$ satisfies (2.1), it follows from the definition of $\mathcal{J}(t)$ that $\mathcal{J}(t)\varphi$ satisfies relation (2.4). Consequently, for any θ, $-r \leq \theta < 0$,

$$\lim_{t \to 0^+} \frac{1}{t}[(\mathcal{J}(t)\varphi(\theta) + \varphi(\theta)] = \frac{d\varphi(\theta)}{d\theta}.$$

If $\theta = 0$, then

$$\lim_{t \to 0^+} \frac{1}{t}[(\mathcal{J}t)\varphi)(0) - \varphi(0)] = \lim_{t \to 0^+} \frac{1}{t}[\int_0^t f(x_\tau)d\tau]$$

$$= f(x_0) = f(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta).$$

But, to say that

$$\mathcal{A}\varphi = \lim_{t \to 0^+} \frac{1}{t}[\mathcal{J}(t)\varphi - \varphi]$$

implies convergence in the norm in $C([-r, 0], R^n)$, which implies uniform convergence and thus $\mathcal{A}\varphi$ must be a continuous function which implies $\mathcal{A}\varphi(0) = f(\varphi) = d\varphi(0)/d\theta$. Summarizing these remarks, we have the following lemma.

<u>Lemma II.3</u>. If $\mathcal{J}(t)$, $t \geq 0$ is the family of transformations on $C([-r, 0], R^n)$ defined by (2.3), then the infinitesimal generator of $\{\mathcal{J}(t)\}$ is given by

$$\varphi(\theta) = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & -h \leq \theta \leq 0 \\ \dfrac{d\varphi(0)}{d\theta} = \displaystyle\int_{-r}^{0} [d\eta(\theta)]\varphi(\theta) = f(\varphi). \end{cases}$$

We now take some results from Hille and Phillips and Reisz-Nagy and apply them to our family of operators $\mathcal{J}(t)$ satisfying properties i) - iv) of Lemma II.1 to obtain the following results.

<u>Lemma II.4</u>. For $t \geq r$, $\sigma(\mathcal{J}(t))$ is a countable set and is a compact set of the complex plane. The only possible point of accumulation of $\sigma(\mathcal{J}(t))$, is $\{0\}$ and if $\mu \neq 0$, μ in $\sigma(\mathcal{J}(t))$, then μ is in $P\sigma(\mathcal{J}(t))$. (Hille and Phillips [8, pp. 180-182], Theorems 5.7.1, 5.7.3]).

Given any operator $\mathcal{J}$, we denote by $\mathcal{N}(\mathcal{J})$ the null space of $\mathcal{J}$; that is, the set of all φ such that $\mathcal{J}\varphi = 0$.

<u>Lemma II.5</u>. For $t \geq r$, if $\mu = \mu(t)$ is in $P\sigma(\mathcal{J}(t))$, $\mu \neq 0$, then for each positive integer k, $\mathcal{N}(\mu I - \mathcal{J}(t)^k$ is of finite dimension for every k, and there exists a least integer n_o such that

$$\mathcal{N}(\mu I - \mathcal{J}(t))^k = \mathcal{N}(\mu I - \mathcal{J}(t))^\ell \quad \text{for all } k, \ell \geq n_o.$$

If $\mathcal{M}_\mu(\mathcal{J}(t))$ is equal to $\mathcal{N}(\mu I - \mathcal{J}(t))^{n_o}$, then

$$\mathcal{J}(t)\mathcal{M}_\mu(\mathcal{J}(t)) \subset \mathcal{M}_\mu(\mathcal{J}(t)).$$

(Hille and Phillips [8, p. 182, Theorem 5.7.3])

<u>Lemma II.6</u>. For $t \geq r$, $P\sigma(\mathcal{J}(t)) = \exp[tP\sigma(\mathcal{A})]$ plus possibly $\{0\}$. More specifically, if $\mu = \mu(t)$ is in $P\sigma(\mathcal{J}(t))$ for some fixed t and $\mu \neq 0$, then there is a point λ in $P\sigma(\mathcal{A})$ such that $e^{\lambda t} = \mu$. Furthermore, if $\{\lambda_n\}$ consists of all distinct points in $P\sigma(\mathcal{A})$ such that

$e^{\lambda_n t} = \mu$, then $\mathcal{N}(\mu I - \mathcal{T}(t))^k$ is the linear extension of the linear independent manifolds $\{\mathcal{N}(\lambda_n I - \mathcal{A})^k\}$. (Hille and Phillips [8, p. 467, Theorem 16.7.2 and pp. 321-324, Theorem 10.6.5]).

In the following, $\mathcal{M}_\lambda(B)$ for any operator B on a Banach space and λ in $P\sigma(B)$ will denote the maximal subspace of $\mathcal{B}$ annihilated by powers of $B - \lambda I$.

Lemma II.7. If $\mathcal{A}$ is the infinitesimal generator of the family of operators defined by (2.3) and λ is in $P\sigma(\mathcal{A})$, then the set $\mathcal{M}_\lambda(\mathcal{A})$ in $C([-r, 0], R^n)$ is finite dimensional. Furthermore, there is a real number β such that $\mathrm{Re}(\lambda) \leq \beta$ for all λ in $P\sigma(\mathcal{A})$, and there are a finite number of λ in $P\sigma(\mathcal{A})$ such that $\gamma \leq \mathrm{Re}\,\lambda$ for any given real number γ.

This lemma is an immediate consequence of Lemmas II.4 - II.6.

Lemma II.6 above gives a very distinctive relationship between the point spectrum of $\mathcal{T}(t)$ and the point spectrum of $\mathcal{A}$. In fact, except for the point $\mu = 0$, $P\sigma(\mathcal{T}(t))$ is completely determined by $P\sigma(\mathcal{A})$. We now derive an explicit expression for $P\sigma(\mathcal{A})$. If λ is in $P\sigma(\mathcal{A})$, then there must exist a nonzero φ in $C([-r, 0], R^n)$ such that

$$\mathcal{A}\varphi = \lambda\varphi.$$

This last relation is satisfied if and only if

$$\frac{d\varphi(\theta)}{d\theta} = \lambda\varphi(\theta), \quad -r \leq \theta \leq 0$$

$$\frac{d\varphi(0)}{d\theta} = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta),$$

which in turn implies $\varphi(\theta) = e^{\lambda\theta}b$, $-r \leq \theta \leq 0$, and the vector b satisfies

$$\text{(2.6)} \qquad \begin{gathered} \Delta(\lambda)b = 0 \\ \Delta(\lambda) = (\lambda I - \int_{-r}^{0} [d\eta(\theta)]e^{\lambda\theta}). \end{gathered}$$

As a result of this fact and Lemma II.7, we have

Lemma II.8. $P\sigma(\mathcal{A}) = \{\lambda \mid \det \Delta(\lambda) = 0\}$. The roots of the characteristic equation of (2.1), $\det \Delta(\lambda) = 0$, have real parts bounded above and there are only a finite number with real parts greater than a given constant.

The characteristic equation of (2.1) can be obtained in a very straightforward manner as in ordinary differential equations by determining necessary and sufficient conditions that the equation (2.1) has a solution of the form $x(t) = e^{\lambda t}b$ for some λ, b. Many procedures have been given to analyze the nature of the roots of the characteristic equation (see, for example, Langer [11], Pontrjagin [14], Pinney [13]). The property of the roots mentioned in Lemma II.3 is also in these papers.

Before proceeding to an analysis of the specific structure of the solutions of (2.1), we wish to derive a result concerning the maximum rate of growth of the solutions of (2.1). For later reference, we state the result independently of solutions of (2.1). If $\mathcal{J}$ is a bounded linear transformation of a Banach space into itself, the spectral radius $\rho_{\mathcal{J}}$ of $\mathcal{J}$ is the smallest closed disk with center at the origin in the complex plane which contains $\sigma(\mathcal{J})$. The following lemma is taken from Reisz-Nagy [15, p. 425].

__Lemma II.9__. If $\mathcal{J}$ is a bounded linear transformation of a Banach space into itself, then the spectral radius $\rho_{\mathcal{J}}$ of $\mathcal{J}$ is given by

$$\rho_{\mathcal{J}} = \lim_{n \to \infty} \|\mathcal{J}^n\|^{1/n}.$$

__Theorem II.1__. If $\mathcal{J}(t)$, $t \geq 0$, is a strongly continuous semigroup of operators of a Banach space $\mathcal{B}$ into itself, if for some $r > 0$, the spectral radius $\rho = \rho_{\mathcal{J}(r)}$ is finite and $\neq 0$ and $\beta r = \log \rho$, then, for any $\gamma > 0$, there is a constant $K(\gamma) \geq 1$ such that

$$\|\mathcal{J}(t)\varphi\| \leq K(\gamma)e^{(\beta+\gamma)t}\|\varphi\| \quad \text{for all} \quad t \geq 0, \quad \varphi \quad \text{in} \quad \mathcal{B}.$$

__Proof__: This proof is essentially the same one as contained in Stokes [19]. Since ρ is finite, the number β is well defined. From Lemma II.9, $e^{\beta r} = \lim_{n \to \infty} \|\mathcal{J}^n(r)\|^{1/n}$, and thus, for any $\gamma > 0$,

$$e^{-\gamma r} = \lim_{n \to \infty} e^{-(\beta+\gamma)r}\|\mathcal{J}^n(r)\|^{1/n}.$$

Therefore, there exists a number N such that

$$e^{-(\beta+\gamma)nr}\|\mathcal{J}^n(r)\| = (e^{-\gamma r} + \epsilon_n)^n$$

where $e^{-\gamma r} + \epsilon_n \leq K < 1$ for all $n \geq N$. Consequently, $e^{-(\beta+\gamma)nr}\|\mathcal{J}^n(r)\| \to 0$ as $n \to \infty$. Since $\mathcal{J}(t)$ is continuous for all $t \geq 0$, there is a constant B such that $\|\mathcal{J}(t)\| \leq B$ for $0 \leq t \leq r$. Define $K(\gamma)$ for any $\gamma > 0$ to be

$$K(\gamma) = \max\{ \max_{0 \le t \le r} Be^{-(\beta+\gamma)t}, \max_{n \ge 0} Be^{-(\beta+\gamma)nr}\|\mathcal{T}^n(r)\|\}.$$

If $0 \le t \le r$, then, for any φ in $\mathcal{B}$,

$$\|\mathcal{T}(t)\varphi\| \le \|\mathcal{T}(t)\|\cdot\|\varphi\| \le B\|\varphi\| \le K(\gamma)e^{(\beta+\gamma)t}\|\varphi\|.$$

If $t \ge r$, then there is an integer n such that $nr \le t < (n+1r)$, and, for all φ in $\mathcal{B}$,

$$\|\mathcal{T}(t)\varphi\| = \|\mathcal{T}(t - nr)\mathcal{T}(nr)\varphi\| \le B\|\mathcal{T}^n(r)\|\cdot\|\varphi\| \le$$

$$[Be^{-(\beta+\gamma)(t-nr)}e^{-(\beta+\gamma)nr}\|\mathcal{T}^n(r)\|]e^{(\beta+\gamma)t}\|\varphi\|$$

$$\le K(\gamma)e^{(\beta+\gamma)t}\|\varphi\|.$$

This completes the proof of the theorem.

Corollary II.1. If $\Delta(\lambda)$ is defined as in (2.7) and all the roots of the characteristic equation $\det \Delta(\lambda) = 0$ satisfy $\operatorname{Re} \lambda \le \beta$, then for any $\gamma \ge 0$, there exists a constant $K(\gamma) \ge 1$ such that if $x(\varphi)$ is the solution of (2.1) with initial function φ in $C([-r, 0], R^n)$ at zero, then

$$\|x_t(\varphi)\| \le K(\gamma)e^{(\beta+\gamma)t}\|\varphi\|, \quad t \ge 0.$$

In particular, if $\beta < 0$, then all solutions of (2.1) approach zero exponentially as $t \to \infty$.

Proof: If $\mathcal{T}(t)$ is defined as in (2.3), then $\rho = \rho_{\mathcal{T}(r)}$ is finite and $= \exp \beta r$ from Lemmas II.6 and II.8 if $\rho \ne 0$. This case follows from Theorem II.1. If $\rho = 0$, the corollary is obviously true.

Corollary II.1 is well known in the literature (see for example, Krasovskii [9], Bellman and Cooke [1], Stokes [19]).

In the following, we shall always assume that $\mathcal{T}(t)$ is the family of operators associated with system (2.1) and $\mathcal{A}$ is the infinitesimal generator of $\mathcal{T}(t)$.

From Lemma II.7, if λ is in $P\sigma(\mathcal{A})$, then $\mathcal{N}(\lambda I - \mathcal{A})^k$ is of finite dimension for every integer k and there is an integer n_o such that $\mathcal{N}(\lambda I - \mathcal{A})^k = \mathcal{N}(\lambda I - \mathcal{A})^{n_o}$ for all $k \geq n_o$. Since $\mathcal{M}_\lambda(\mathcal{A}) = \mathcal{N}(\mathcal{A} - \lambda I)^{n_o}$ has finite dimension, say d, there exists a basis $\varphi_1, \ldots, \varphi_d$ of $\mathcal{M}_\lambda(\mathcal{A})$. Furthermore, since $\mathcal{A}\mathcal{M}_\lambda(\mathcal{A}) \subset \mathcal{M}_\lambda(\mathcal{A})$, there exists a $d \times d$ constant matrix B (whose only eigenvalue is λ) such that

$$\mathcal{A}\Phi = \Phi B, \quad \Phi = \mathrm{row}(\varphi_1, \ldots, \varphi_d).$$

Now, let us investigate the nature of the solutions of (2.1) given by $\mathcal{T}(t)\Phi$. From Lemma II.2,

$$\frac{d}{dt}[\mathcal{T}(t)\Phi] = \mathcal{T}(t)\mathcal{A}\Phi = \mathcal{T}(t)\Phi B$$

and, thus,

$$\mathcal{T}(t)\Phi = \Phi e^{Bt}.$$

Consequently, we have the following result.

<u>Theorem II.2.</u> Let $\mathcal{T}(t)$, $t \geq 0$, be the strongly continuous semigroup of operators defined by (2.3) and let $\mathcal{A}$ be the infinitesimal generator

of $\mathcal{T}(t)$. Suppose λ is in $P\sigma(\mathcal{A})$ and let $\Phi = \mathrm{row}(\varphi_1, \ldots, \varphi_d)$ be a basis for $\mathcal{M}_\lambda(\mathcal{A})$, the maximal subspace of $C([-r,0], R^n)$ annihilated by powers of $\lambda I - \mathcal{A}$. If a is an arbitrary constant column vector of dimension d, then $\mathcal{T}(t)\Phi a = \Phi e^{Bt}a$ for some constant matrix B with all of its eigenvalues equal to λ. If $\mu = \mu(t) \neq 0$ is in $P\sigma(\mathcal{T}(t)$ for some t, and $\{\lambda_1, \ldots, \lambda_p\}$ is the set of distinct elements in $P\sigma(\mathcal{A})$ such that $e^{\lambda t} = \mu$, and $\Phi = (\Phi_1, \ldots, \Phi_p)$, where Φ_j is a basis for $\mathcal{M}_{\lambda_j}(\mathcal{A})$, of dimension d_j then, for any $\varphi \in \mathcal{M}_\mu(\mathcal{T}(t))$, there exists a vector a of dimension $d = d_1 + \ldots + d_p$ such that $\varphi = \Phi a$ and

$$\mathcal{T}(t)\varphi = \Phi\, e^{Bt}a \tag{2.7}$$

where B is a $d \times d$ matrix given by $B = \mathrm{diag}\,(B_1, \ldots, B_p)$ where each B_j is defined by $\mathcal{A}\Phi_j = \Phi_j B_j$.

The first part of this theorem has been proved above and the second part follows by using the same argument together with Lemma II.6.

Theorem II.2 shows that on the subspace $\mathcal{M}_\mu(\mathcal{T}(t))$, the functional-differential equation (2.1) has the same structure as an ordinary linear differential with constant coefficients. Of course, the dimension of the matrix B in (2.7) may have nothing to do with the dimension of system (2.1) since the dimension of B is determined by the multiplicity of the eigenvalue $\mu(t)$ of $\mathcal{T}(t)$. Notice also that the multiplicity of this eigenvalue may change with t. In fact, this is easily illustrated by an ordinary equation

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} 2\pi i & 0 \\ 0 & -2\pi i \end{pmatrix}.$$

Then $\mathcal{J}(t) = e^{At}$ and $P\sigma(\mathcal{J}(t)) = (e^{2\pi it}, e^{-2\pi it})$. For $t = n$ an integer, $P\sigma(\mathcal{J}(n) = \{1\}$ and, otherwise, consists of two distinct points. On the other hand, the multiplicity of the points in $P\sigma(\mathcal{Q})$ do not change and one can always define the set $\mathcal{M}_\lambda(\mathcal{Q})$ to generate solutions of (2.1).

III. The adjoint equation and a change of coordinates. In this section, we wish to show how one can introduce a change of coordinates in the space $C([-r, 0], R^n)$ in such a way as to exhibit in a natural manner any particular eigenspace $\mathcal{M}_\lambda(\mathcal{A})$ associated with system (2.1). More specifically, we show that it is possible to transform system (2.1) to an equivalent system which consists of a set of ordinary differential equations (whose solutions describe the behavior of $\mathcal{T}(t)$ on the eigenspace $\mathcal{M}_\lambda(\mathcal{A})$) together with an operator equation (whose solutions describe the behavior of $\mathcal{T}(t)$ on a space complementary to $\mathcal{M}_\lambda(\mathcal{A})$). To do this, we make use of the equation "adjoint" to (2.1). This concept has been used in functional-differential equations by many authors (see, for example, de Bruijn [4], Bellman and Cooke [2], Hahn [5], Halanay [6] and Shimanov [17,18]) The author has been influenced by all of these authors, but especially by Shimanov.

Consider the equation

$$\dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t + \theta), \quad t \geq 0 \tag{3.1}$$

and its formal "adjoint"

$$\dot{y}(s) = - \int_{-r}^{0} [d\eta^T(\theta)]y(s - \theta), \quad s \leq 0 \tag{3.2}$$

where x, y are n-vectors and the symbol B^T always denotes the transpose of a matrix B. The term adjoint is justified by the following observation. If we let $Lx(t) = \dot{x}(t) - \int_{-r}^{0} [d\eta(\theta)]x(t + \theta)$, $L^*y(s) = \dot{y}(s) + \int_{-r}^{0} [d\eta^T(\theta)]y(s - \theta)$ be operators defined on differentiable functions, then

$$y^T(t)(Lx)(t) + [(L^*y)(t)]^T x(t) =$$

$$\frac{d}{dt}[y^T(t)x(t) - \int_{-r}^{0}\int_{0}^{\theta} y^T(t+\xi-\theta)[d\eta(\theta)]x(t+\xi)d\xi].$$

The expression on the right hand side of this equation will play an important role in the following discussion and, thus, we give it a special designation. For any $\varphi \in C([-r, 0], R^n)$, $\psi \in C([0, r], R^n)$, we define the symbol (ψ, φ) by the relation

$$(3.3) \qquad (\psi, \varphi) = \psi^T(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi^T(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi.$$

For the solution $x(\varphi)$ of (3.1) with initial value φ in $C([-r, 0], R^n)$ at zero, we have already defined the operator $\mathcal{J}(t)$, $t \geq 0$, by the relation $x_t(\varphi) = \mathcal{J}(t)\varphi$ and the infinitesimal generator $\mathcal{A}$ of $\mathcal{J}(t)$ was

$$(3.4) \qquad \varphi(\theta) = \begin{cases} \dot{\varphi}(\theta), & -r \leq \theta \leq 0 \\ \dot{\varphi}(0) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta). \end{cases}$$

If ψ is in $C([0, r], R^n)$, then system (3.2) has a solution $y(\psi)$ with initial function ψ at zero and defined for all $s \leq r$. If we let $y_s(\psi)(\theta) \stackrel{\text{def}}{=} y(\psi)(s+\theta)$, $0 \leq \theta \leq r$, $s \leq 0$, then the operator $\mathcal{J}^*(s)$, $s \leq 0$, defined by $y_s(\psi) = \mathcal{J}^*(s)\psi$ has all of the same properties as $\mathcal{J}(t)$. The infinitesimal generator $\mathcal{A}_1^*$ of $\mathcal{J}^*(s)$ is defined by

$$\mathcal{A}_1^*\psi = \lim_{s \to 0^-} \frac{1}{s}[\mathcal{J}^*(s)\psi - \psi].$$

For later purposes, it is convenient to define an operator $\mathcal{A}^*$ on $C([0, r], R^n)$ in such a way that $\mathcal{A}^* = -\mathcal{A}_1^*$. A direct calculation shows that

$$(3.5) \qquad \mathcal{A}^*\psi(\theta) = \begin{cases} -\dot{\psi}(\theta), & 0 \leq \theta \leq r \\ -\dot{\psi}(0) = \int_{-r}^{0} [d\eta^T(\theta)]\psi(-\theta). \end{cases}$$

It is also easy to verify that $\mathcal{J}^*(s)$ and $\mathcal{A}^*$ satisfy the following relationship:

$$(3.6) \qquad \frac{d\mathcal{J}^*(s)}{ds} = -\mathcal{A}^*\mathcal{J}^*(s) = -\mathcal{J}^*(s)\mathcal{A}^*, \quad s \leq 0.$$

It is obvious that $x(t)$, $y(t)$ continuously differentiable n-vector functions and x_t, y_t defined by $x_t(\theta) = x(t + \theta)$, $y_t(\theta) = y(t - \theta)$, $-h \leq \theta \leq 0$, implies

$$\frac{d}{dt}(y_t, x_t) = (\dot{y}_t, x_t) + (y_t, \dot{x}_t),$$

where $\dot{y}_t(\theta) = dy(t - \theta)/dt = \dot{y}(t - \theta)$, $\dot{x}_t(\theta) = dx(t + \theta)/d\theta = \dot{x}(t + \theta)$, $-h \leq \theta \leq 0$.

Lemma III.1. $(\psi, \mathcal{A}\varphi) = (\mathcal{A}^*\psi, \varphi)$ for all φ in $\mathcal{D}(\mathcal{A})$, ψ in $\mathcal{D}(\mathcal{A}^*)$.

Proof:
$$(\psi, \mathcal{A}\varphi) = \psi^T(0)\Phi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi^T(\xi-\theta)[d\eta(\theta)]\Phi(\xi)d\xi$$

$$= \psi^T(0)\Phi(0) - \int_{-r}^{0} [\psi^T(\xi-\theta)(d\eta(\theta)\varphi(\xi)]_0^{\theta}$$

$$+ \int_{-r}^{0}\int_{0}^{\theta} \frac{d\psi^T(\xi-\theta)}{d\xi}[d\eta(\theta)]\varphi(\xi)d\xi$$

$$= \int_{-r}^{0} \psi^T(-\theta)(d\eta(\theta))\varphi(0) + \int_{-r}^{0}\int_{0}^{\theta} \frac{d\psi^T(\xi-\theta)}{d\xi}[d\eta(\theta)]\varphi(\xi)d\xi$$

$$= -\dot{\psi}^T(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} [-\frac{d\psi^T(\xi-\theta)}{d\xi}]d\eta(\theta)\varphi(\xi)d\xi$$

$$= (\mathcal{A}^*\psi, \varphi)$$

as was to be shown.

Lemma III.2. If $\tau \geq 0$ is any given constant and $\varphi \in C([-r, 0], R^n)$, $\psi \in C([\tau, \tau + r], R^n)$, then

$$(\mathcal{J}^*(t-\tau)\psi, \mathcal{J}(t)\varphi) = \text{constant for } 0 \leq t \leq \tau.$$

Proof: From the properties of $\mathcal{J}^*$ and $\mathcal{J}$, it follows from Lemma III.1 that

$$\frac{d}{dt}(\mathcal{J}^*(t-\tau)\psi, \mathcal{J}(t)\varphi) = -(\mathcal{A}^*\mathcal{J}^*(t-\tau)\psi, \mathcal{J}(t)\varphi) + (\mathcal{J}^*(t-\tau)\psi, \mathcal{A}\mathcal{J}(t)\varphi) = 0.$$

Lemma III.3. If $\lambda \neq \mu$ then for any nonnegative integers k, ℓ, $\psi \in \mathcal{N}(\mathcal{A}^* - \mu I)^{k+1}$, and $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^{\ell+1}$ implies

$$((\mathcal{A}^* - \mu I)^{k-p}\psi, (\mathcal{A} - \lambda I)^{\ell-q}\varphi) = 0, \quad 0 \leq p \leq k, \quad 0 \leq q \leq \ell.$$

Proof: We always suppose $\lambda \neq \mu$. Suppose $k = 0$. We wish to show that that for any nonnegative integer ℓ and any q, $0 \leq q \leq \ell$,

$$(3.7) \qquad (\psi, (\mathcal{A} - \lambda I)^{\ell-q}\varphi) = 0.$$

We prove this by induction on ℓ. If $\ell = 0$, then $q = 0$ and $\mathcal{A}^*\psi = \mu\psi$, $\mathcal{A}\varphi = \lambda\varphi$ which implies

$$(\psi, \mathcal{A}\varphi) = \lambda(\psi, \varphi) = (\mathcal{A}^*\psi, \varphi) = \mu(\psi, \varphi)$$

and thus $(\psi, \varphi) = 0$. If we suppose (3.7) has been proven for $0 \leq \ell \leq \gamma - 1$, then for $\psi \in \mathcal{N}(\mathcal{A}^* - \mu I)$, $(\mathcal{A} - \lambda I)^{\gamma-q}\varphi \in \mathcal{N}(\mathcal{A} - \lambda)^{q+1}$, the induction hypothesis implies

$$(3.8) \qquad (\psi, (\mathcal{A} - \lambda I)^{\gamma-q}\varphi) = 0 \quad \text{for } 0 \leq q \leq \gamma - 1.$$

Furthermore, for $q = \gamma - 1$,

$$0 = (\psi, (\mathcal{A} - \lambda I)\varphi) = -\lambda(\psi, \varphi) + (\mathcal{A}^*\psi, \varphi) = (-\lambda + \mu)(\psi, \varphi)$$

which implies $(\psi, \varphi) = 0$. But this is (3.8) for $q = \gamma$. Consequently, relation (3.7) is true.

Now let us suppose the conditions of the lemma are satisfied for all $k \leq \gamma - 1$ and all integers ℓ. We wish to show this is true for $k = \gamma$. Since for $k = \gamma$ and any ℓ, $\psi \in \mathcal{N}(\mathcal{A}^* - \mu I)^{k+1}$, $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^{\ell+1}$ implies $(\mathcal{A}^* - \mu I)^{\gamma-p}\psi \in \mathcal{N}(\mathcal{A}^* - \mu I)^{p+1}$, $(\mathcal{A} - \lambda I)^{\ell-q}\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^{q+1}$, for $0 \leq p \leq \gamma$, $0 \leq q \leq \ell$, the induction hypothesis implies that

$$(3.9) \qquad ((\mathcal{A}^* - \mu I)^{\gamma-p}\psi, (\mathcal{A} - \lambda I)^{\ell-q}\varphi) = 0, \quad 0 \leq p \leq \gamma - 1, \; 0 \leq q \leq \ell.$$

It remains to show these relations are true for $p = \gamma$; that is,

$$(3.10) \qquad (\psi, (\mathcal{A}-\lambda I)^{\ell-q}\varphi) = 0, \quad 0 \leq q \leq \ell.$$

Let us first show that $(\psi, (\mathcal{A}-\lambda I)^{\ell}\varphi) = 0$. We know from (3.9) that

$0 = ((\mathcal{A}^* - \mu I)\psi, (\mathcal{A} - \lambda I)^{\ell}\varphi) = (\psi, (\mathcal{A} -\mu I)(\mathcal{A}-\lambda I)^{\ell}\varphi) = (\lambda-\mu)(\psi, (\mathcal{A} -\lambda I)^{\ell}\varphi)$ since $\mathcal{A} (\mathcal{A}-\lambda I)^{\ell}\varphi = \lambda(\mathcal{A} -\lambda I)^{\ell}\varphi$. This is the desired relation. Consequently, (3.10) is true for $q = 0$. Suppose (3.10) has been shown to be true for all $q \leq \eta - 1$. From (3.9),

$$0 = ((\mathcal{A}^* -\mu I)\psi, (\mathcal{A}-\lambda I)^{\ell-\eta}\varphi) = (\psi, (\mathcal{A}-\lambda I + (\lambda-\mu)I)(\mathcal{A}-\lambda I)^{\ell-\eta}\varphi) =$$

$$= (\psi, (\mathcal{A}-\lambda I)^{\ell-(\eta-1)}\varphi) + (\lambda - \mu)(\psi, (\mathcal{A} -\lambda I)^{\ell-\eta}\varphi)$$

$$= (\lambda - \mu)(\psi, (\mathcal{A} -\lambda I)^{\ell-\eta}\varphi)$$

by the induction hypothesis. Consequently, $(\psi, (\mathcal{A} -\lambda I)^{\ell-\eta}\varphi) = 0$ and (3.10) is valid for all q. This completes the proof of the lemma.

Corollary III.1. If $\psi \in \mathfrak{M}_{\mu}(\mathcal{A}^*)$, $\varphi \in \mathfrak{M}_{\lambda}(\mathcal{A})$, then $(\psi, \varphi) = 0$ if $\lambda \neq \mu$.

Proof: Take $p = k$, $q = \ell$ in the above lemma.

Lemma III.4. If $\psi \in \mathfrak{N}(\mathcal{A}^* - \lambda I)$, then for all $\varphi \in \mathcal{D}(\mathcal{A})$

$$(\psi, (\mathcal{A} -\lambda I)\varphi) = 0;$$

that is, $\mathcal{N}(\mathcal{A}^* - \lambda I)$ is orthogonal to the range, $\mathcal{R}(\mathcal{A} - \lambda I)$, of $\mathcal{A} - \lambda I$, the orthogonality being with respect to the symbol (ψ, φ). If $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)$, then for all $\psi \in \mathcal{D}(\mathcal{A}^*)$,

$$((\mathcal{A}^* - \lambda I)\psi, \varphi) = 0.$$

Proof: This is an immediate consequence of Lemma III.1.

Lemma III.5. λ is in $P\sigma(\mathcal{A})$ if and only if λ is in $P\sigma(\mathcal{A}^*)$.

Proof: λ is in $P\sigma(\mathcal{A})$ if and only if $\det \Delta(\lambda) = 0$ where $\Delta(\lambda)$ is defined in (2.6). λ is in $P\sigma(\mathcal{A}^*)$ if and only if

$$\psi(\theta) = e^{-\lambda\theta} b$$

and b is a nonzero solution of the equation,

$$[\lambda I - \int_{-r}^{0} [d\eta^T(s)] e^{\lambda s}] b = \Delta^T(\lambda) b = 0,$$

or if and only if $\det \Delta^T(\lambda) = 0$. But $\det \Delta^T(\lambda) = \det \Delta(\lambda)$ and the lemma is proved.

Lemma III.1, Corollary III.1, Lemma III.4 and Lemma III.5 indicate that the operators $\mathcal{A}$ and $\mathcal{A}^*$ are in some sense adjoint to each other; namely, $(\psi, \mathcal{A}\varphi) = (\mathcal{A}^*\psi, \varphi)$, $\mathcal{A}^*$ and $\mathcal{A}$ have the same point spectrum, $\mathcal{M}_\mu(\mathcal{A}^*)$ is orthogonal to $\mathcal{M}_\lambda(\mathcal{A})$ if $\lambda \neq \mu$ and $\mathcal{N}(\mathcal{A}^* - \lambda I)$ is orthogonal to $\mathcal{R}(\mathcal{A} - \lambda I)$. If we could show that $\mathcal{M}_\lambda(\mathcal{A})$ and $\mathcal{M}_\lambda(\mathcal{A}^*)$ have the same dimension and the matrix formed by (ψ_j, φ_k), where the ψ_j, φ_k are bases for $\mathcal{M}_\lambda(\mathcal{A}^*)$ $\mathcal{M}_\lambda(\mathcal{A})$, respectively, is nonsingular, then there is

almost a complete similarity with the usual concept of adjoint. The next results are a precise statement of these remarks.

Lemma III.6. If λ is in $P\sigma(\mathcal{A})$, then the dimensions of $\mathcal{N}_\lambda(\mathcal{A})$ and $\mathcal{N}_\lambda(\mathcal{A}^*)$ are equal, $\mathcal{N}_\lambda(\mathcal{A}^*) = \mathcal{N}((\mathcal{A}^* - \lambda I)^k$, $\mathcal{N}_\lambda(\mathcal{A}) = \mathcal{N}(\mathcal{A} - \lambda I)^k$, for some k with $\mathcal{N}(\mathcal{A}^* - \lambda I)^k \neq \mathcal{N}((\mathcal{A}^* - \lambda I)^{k-1}$, $\mathcal{N}(\mathcal{A} - \lambda I)^k \neq \mathcal{N}((\mathcal{A} - \lambda I)^k$. If $\varphi \in \mathcal{N}_\lambda(\mathcal{A})$, then a necessary and sufficient condition that $\varphi \in \mathcal{R}((\mathcal{A} - \lambda I)^k$ is that

$$(\psi, \varphi) = 0 \quad \text{for all} \quad \psi \in \mathcal{N}((\mathcal{A}^* - \lambda I)^k.$$

Proof: First, we introduce some notation. With the matrix $\Delta(\lambda)$ defined in (2.6), we define matrices $P_j(\lambda)$ as

$$P_{j+1}(\lambda) = \frac{\Delta^{(j)}(\lambda)}{j!}, \quad \Delta^{(j)}(\lambda) = \frac{d^j}{d\lambda^j}(\Delta(\lambda)), \quad j = 0,1,2, \ldots$$

and the matrix A_k of dimension $(kn)\times(kn)$ as

$$A_k = \begin{bmatrix} P_1 & P_2 & \cdots & P_k \\ 0 & P_1 & \cdots & P_{k-1} \\ \vdots & & & \\ 0 & 0 & \cdots & P_1 \end{bmatrix}.$$

Let $\alpha_k = \text{col}\,(\alpha_{k1}, \ldots, \alpha_{kk})$, $\beta_k = \text{col}\,(\beta_{k1}, \ldots, \beta_{kk})$, where each α_{kj}, β_{kj} is an $n \times p$ matrix, be bases for $\mathcal{N}(A_k)$, $\mathcal{N}(A_k^T)$, respectively.

Suppose $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^k$. Then it is easy to show that

$$\varphi(\theta) = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta}$$

where $A_k\gamma = 0$, $\gamma = \mathrm{col}\,(\gamma_1, \ldots, \gamma_k)$; that is, the vector γ must belong to the null space of A_k. Consequently, a basis Φ_k for $\mathcal{N}(\mathcal{A} - \lambda I)^k$ can be written as

$$\Phi_k(\theta) = \sum_{j=0}^{k=1} \alpha_{k,j+1} \frac{\theta^j}{j!} e^{-\lambda\theta}.$$

In the same way, one shows that a basis Ψ_k for $\mathcal{N}(\mathcal{A}^* - \lambda I)^k$ can be written as

$$\Psi_k(\theta) = \sum_{j=0}^{k-1} \beta_{k,k-j} \frac{(-\theta)^j}{j!} e^{-\lambda\theta}.$$

These remarks prove that the dimensions of $\mathcal{N}(\mathcal{A} - \lambda I)^k$ and $\mathcal{N}(\mathcal{A}^* - \lambda I)^k$ are the same for every value of k and the first part of Lemma III.6 follows immediately.

Now, suppose that $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^\ell$ and $\psi \in \mathcal{N}(\mathcal{A}^* - \lambda I)^k$. Then $\psi(\theta) = \Psi_k(\theta)b$, $\varphi(\theta) = \theta_\ell(\theta)a$, for some constant vectors b,a, and

$$\begin{aligned}(\psi, \varphi) &= b^T \Psi_k^T(0)\Phi_\ell(0)a - \int_{-r}^{0} \int_{0}^{\theta} b^T \Psi_k^T(\xi-\theta)d\eta(\theta)\Phi_\ell(\xi)a\, d\xi \\ &= b^T [\Psi_k^T(0)\Phi_\ell(0) - \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \beta_{k,k-i}^T(\int_{-r}^{0} e^{\lambda\theta}d\eta(\theta) \int_0^\theta \frac{(\theta-\xi)^i \xi^j}{i!\, j!} d\xi)\alpha_{\ell,j+1}]a \\ &= b^T [\beta_{kk}^T\alpha_{kl} - \sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \beta_{k,k-i}^T(\int_{-r}^{0} \frac{\theta^{i+j+1}}{(i+j+1)!} e^{\lambda\theta}d\eta(\theta))\alpha_{\ell,j+1}]a.\end{aligned}$$

Consequently,

$$(3.11)\qquad (\psi, \varphi) = b^T \left[\sum_{i=0}^{k-1} \sum_{j=0}^{\ell-1} \beta^T_{k,k-i} P_{i+j+2} \alpha_{\ell,j+1} \right] a, \quad \psi = \Psi_k b, \ \varphi = \Phi_\ell a.$$

For any $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^\ell$, we wish to derive necessary and sufficient conditions that $\varphi \in \mathcal{R}(\mathcal{A} - \lambda I)^k$; that is, there exists a φ_* such that

$$(3.12)\qquad (\mathcal{A} - \lambda I)^k \varphi_* = \varphi.$$

If $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^\ell$, there exists an a such that

$$(3.13)\qquad \begin{aligned} \varphi(\theta) &= \Phi_\ell(\theta) a = \sum_{j=0}^{\ell-1} \alpha_{\ell,j+1} a \frac{\theta^j}{j!} e^{\lambda\theta} = \sum_{j=0}^{\ell-1} \gamma_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta}, \\ \gamma_{j+1} &= \alpha_{\ell,j+1} a, \quad j = 0,1, \ldots, \ell-1. \end{aligned}$$

If (3.12) is to have a solution. then necessarily

$$\left(\frac{d}{d\theta} - \lambda\right)^k \varphi_*(\theta) = \varphi(\theta)$$

or

$$(3.14)\qquad \varphi_*(\theta) = \sum_{j=0}^{k-1} \gamma^*_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta} + \sum_{j=0}^{\ell-1} \gamma_{j+1} \frac{\theta^{k+j}}{(k+j)!} e^{\lambda\theta},$$

where the γ^*_{j+1} are n-vectors satisfying some additional conditions which we proceed to derive.

If we define $\varphi_*^{(m)}(\theta)$ to be equal to $(d/d\theta - \lambda)^m \varphi_*(\theta)$ with φ_* given in (3.14), then

$$\varphi_*^{(m)}(\theta) = \sum_{j=0}^{k-m-1} \gamma^*_{m+j+1} \frac{\theta^j}{j!} e^{\lambda\theta} + \sum_{j=0}^{\ell-1} \gamma_{j+1} \frac{\theta^{k+j-m}}{(k+j-m)!} e^{\lambda\theta},$$

$$m = 0,1,2, \ldots, k-1.$$

If φ_* is to satisfy (3.12), then $\varphi_*^{(m)}$ must be in $\mathcal{D}(\mathcal{A} - \lambda I)$ for $m = 0,1,2, \ldots, k-1$; that is,

$$\phi_*^{(m)}(0) = \int_{-r}^{0} d\eta(\theta)\varphi_*^{(m)}(\theta)$$

or,

$$\lambda\gamma^*_{m+1} + \gamma^*_{m+2} = \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)\Big(\sum_{j=0}^{k-m-1} \gamma^*_{m+j+1} \frac{\theta^j}{j!} + \sum_{j=0}^{\ell-1} \gamma_{j+1} \frac{\theta^{k+j-m}}{(k+j-m)!}\Big),$$

$$m = 0,1,2, \ldots, k-1.$$

Using the definition of the matrices P_j above, these last equations can be written as

$$P_1\gamma_{m+1} + P_2\gamma_{m+2} + \ldots + P_{k-m}\gamma_k = \sum_{j=0}^{\ell=1} P_{k+j-m+1}\gamma_{j+1}.$$

Letting

$$\gamma^* = \operatorname{col}\,(\gamma_1^*, \ldots, \gamma_k^*)$$

$$d = \operatorname{col}\,(d_1, \ldots, d_k)$$

$$d_s = \sum_{j=0}^{\ell-1} P_{k+j-s+2}\gamma_{j+1} = \sum_{j=0}^{\ell-1} P_{k+j-s+2}\alpha_{\ell,j+1}a,$$

we have the following result: a necessary and sufficient condition that $\varphi \in \mathcal{N}(\mathcal{A} - \lambda I)^\ell$ implies $\varphi \in \mathcal{R}(\mathcal{A} - \lambda I)^k$ is that the set of equations

$$A_k \gamma^* = d$$

have a solution; that is, $e^T d = 0$ for all elements $e \in \mathcal{N}(A_k^T)$. But, for any $e \in \mathcal{N}(A_k^T)$, there exists a b such that $e = \beta_k b$, so that the desired property holds if and only if $b^T \beta_k^T d = 0$ for all p-vectors b, or, if and only if,

$$\begin{aligned} 0 &= b^T \beta_k^T d \\ &= b^T \left(\sum_{s=1}^{k} \beta_{ks}^T \sum_{j=0}^{\ell-1} P_{k+j-s+2} \alpha_{\ell, j+1} \right) a \\ &= b^T \left(\sum_{s=1}^{k} \sum_{j=0}^{\ell-1} \beta_{k,\, k-i}^T P_{i+j+2} \alpha_{\ell, j+1} \right) a \end{aligned}$$

for all b. But, from (3.11), this latter relation is true if and only if $(\psi, \varphi) = 0$ for all $\psi \in \mathcal{N}(\mathcal{A}^* - \lambda I)^k$.

This completes the proof of the lemma.

Lemma III.7. If Ψ is a basis for $\mathcal{M}_\lambda(\mathcal{A}^*)$ and Φ is a basis for $\mathcal{M}_\lambda(\mathcal{A})$, then (Ψ, Φ) is nonsingular and may be taken as the identity. Furthermore, if $(\Psi, \Phi) = I$ and the square matrices B^*, B are defined by $\mathcal{A}^* \Psi = \Psi B^*$, $\mathcal{A} \Phi = \Phi B$, then $B^* = B^T$.

Proof: Suppose $\mathcal{M}_\lambda(\mathcal{A}^*) = \mathcal{N}(\mathcal{A}^* - \lambda I)^k$, $\mathcal{M}_\lambda(\mathcal{A}) = \mathcal{N}(\mathcal{A} - \lambda I)^k$ where k is the least integer for which this is true and Ψ, Φ are bases for $\mathcal{M}_\lambda(\mathcal{A}^*)$, $\mathcal{M}_\lambda(\mathcal{A})$, respectively. If there exists a constant vector a such that $(\Psi, \Phi)a = (\Psi, \Phi a) = 0$, then, for $\varphi = \Phi a$, we have $(\psi, \varphi) = 0$ for all $\psi \in \mathcal{N}(\mathcal{A}^* - \lambda I)^k$. Consequently, from Lemma III.6, $\varphi \in \mathcal{R}(\mathcal{A} - \lambda I)^k$. Thus, there is a φ_* such that $\varphi = (\mathcal{A} - \lambda I)^k \varphi_*$. Obviously, $\varphi_* \in \mathcal{M}_\lambda(\mathcal{A})$ and $\mathcal{M}_\lambda(\mathcal{A}) = \mathcal{N}(\mathcal{A} - \lambda I)^k$ implies

$\varphi = (\mathcal{A} - \lambda I)^k \varphi_* = 0$. Finally, $0 = \varphi = \Phi a$ implies $a = 0$ since Φ is a basis for $\mathcal{M}_\lambda(\mathcal{A})$ and the matrix (Ψ, Φ) is nonsingular. By a change of basis one can obviously take $(\Psi, \Phi) = I$. From Lemma III.1 and the definition of (ψ, φ), we have

$$\begin{aligned}(\Psi, \mathcal{A}\Phi) &= (\Psi, \Phi B) = (\Psi, \Phi)B = B \\ &= (\mathcal{A}^*\Psi, \Phi) = (\Psi B^*, \Phi) = B^{*T} (\Psi, \Phi) = B^{*T},\end{aligned}$$

which completes the proof of the lemma.

If one does not choose $(\Psi, \Phi) = I$, then the matrices B^* and B are related by $B = (\Psi, \Phi)^{-1} B^{*T} (\Psi, \Phi)$.

Lemma III.8. If λ is in $P\sigma(\mathcal{A})$ and Φ is a basis for $\mathcal{M}_\lambda(\mathcal{A})$, then the solution $x(\varphi)$ of (3.1) with initial function φ in $\mathcal{M}_\lambda(\mathcal{A})$ at zero, is defined for all values of t in $(-\infty, \infty)$ and

$$x(\varphi)(t) = \Phi(0)e^{Bt}, \qquad -\infty < t < \infty,$$

where B is the matrix defined by $\mathcal{A}\Phi = \Phi B$. If Ψ is a basis for $\mathcal{M}_\lambda(\mathcal{A}^*)$, $(\Psi, \Phi) = I$, then the solution $y(\psi)$ of (3.2) with initial function ψ in $\mathcal{M}_\lambda(\mathcal{A}^*)$ is defined for all t in $(-\infty, \infty)$ and

$$y(\psi)(t) = \Psi(0)e^{-B^T t}, \quad -\infty < t < \infty$$

where B^T is the matrix defined by $\mathcal{A}^*\Psi = \Psi B^T$.

The proof of this is obvious.

Let $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$ be defined as before, have dimension p, and let Φ, Ψ be bases of $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$, respectively. From Lemma III.7, we can choose $(\Psi, \Phi) = I$, the identity. For any

$\varphi \in C([-r, 0], R^n)$, we define a vector $b \in R^p$ and a function $\bar{\varphi} \in C([-r,0],R^n)$ by the relation

$$\varphi = \Phi b + \bar{\varphi}, \qquad b = (\Psi, \varphi). \tag{3.15}$$

It follows that $(\Psi, \bar{\varphi}) = 0$ and, also, this decomposition is unique.

If we define $\dot{x}_t$ by $\dot{x}_t(\theta) = \dot{x}(t + \theta)$, $-r \leq \theta \leq 0$, then the equation (3.1) can be written as

$$\dot{x}_t = \mathcal{A} x_t, \qquad t \geq 0. \tag{3.16}$$

For any solution $x_t = x_t(\varphi)$, φ in $C([-r, 0], R^n)$, consider the change of variables

$$x_t = \Phi y(t) + z_t, \quad t \geq 0, \quad y(t) = (\Psi, x_t). \tag{3.17}$$

Since $(\Psi, z_t) = 0$ for all t, this implies

$$\dot{y}(t) = \frac{d}{dt}(\Psi, x_t) = (\Psi, \mathcal{A} x_t) = B(\Psi, x_t) = By(t)$$

$$\dot{z}_t = \dot{x}_t - \Phi\dot{y}(t) = \mathcal{A} x_t - \Phi By(t) = \mathcal{A}(x_t - \Phi y(t)) = \mathcal{A} z_t,$$

$$(\Psi, z_t) = 0.$$

Consequently, we have the following

<u>Theorem III.1.</u> Suppose the operators $\mathcal{A}$, $\mathcal{A}^*$ are defined by (3.4), (3.5), respectively, and let $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$, $\lambda \in P\sigma(\mathcal{A})$, be the maximal subspaces of $C([-r, 0], R^n)$, $C([0, r], R^n)$ annihilated by powers of $(\mathcal{A} - \lambda I)$, $(\mathcal{A}^* - \lambda I)$, respectively. If Φ is a basis for $\mathcal{M}_\lambda(\mathcal{A})$, Ψ is a basis for $\mathcal{M}_\lambda(\mathcal{A}^*)$, $(\Psi, \Phi) = I$, the identity, where

(Ψ, φ) is defined in (3.3), then the change of variables (3.17) applied to (3.16) yields the equivalent system

$$\text{(3.18)} \qquad \begin{aligned} \dot{y}(t) &= B\, y(t) \\ \dot{z}(t) &= \mathcal{A} z_t, \quad (\Psi, z_t) = 0 \end{aligned}$$

where B is a matrix defined by $\mathcal{A}\Phi = \Phi B$, has all of its eigenvalues equal to λ, and the point spectrum of the operator $\mathcal{A}$ restricted to the set of $\bar{\varphi}$ such that $(\Psi, \bar{\varphi}) = 0$ does not contain λ.

Now let us consider the perturbed equation

$$\text{(3.19)} \qquad \begin{aligned} \dot{x}(t) &= f(x_t) + G(t, x_t), \\ f(\varphi) &= \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta), \end{aligned}$$

where G is some function defined for $0 \leq t \leq \infty$, $\varphi \in C([-r, 0], R^n)$. If we define the operator $\mathcal{A}$ as in (3.4) and the operator $\mathcal{G}_t$ by the relation

$$\text{(3.20)} \qquad \mathcal{G}_t\varphi(\theta) = \begin{cases} 0 \quad , & -r \leq \theta < 0, \\ G(t, \varphi), & \theta = 0, \end{cases}$$

then system (3.19) is equivalent to

$$\text{(3.21)} \qquad \dot{x}_t = \mathcal{A} x_t + \mathcal{G}_t x_t.$$

With Φ, Ψ defined as before, $(\Psi, \Phi) = I$, the transformation (3.17) applied to (3.21) yields

$$
\begin{aligned}
\dot{y}(t) &= (\Psi, \dot{x}_t) = (\Psi, \mathcal{A}x_t) + (\Psi, \mathcal{G}_t x_t) \\
&= (\mathcal{A}^*\Psi, x_t) + (\Psi, \mathcal{G}_t x_t) \\
&= B(\Psi, x_t) + (\Psi, \mathcal{G}_t x_t) \\
&= By(t) + (\Psi, \mathcal{G}_t x_t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_t &= \dot{x}_t - \Phi\dot{y}(t) = \mathcal{A}x_t + \mathcal{G}_t x_t - \Phi By(t) - \Phi(\Psi, \mathcal{G}_t x_t) \\
&= \mathcal{A}(x_t - \Phi y(t)) + \mathcal{G}_t x_t - \Phi(\Psi, \mathcal{G}_t x_t) \\
&= \mathcal{A}z_t + \mathcal{G}_t x_t - \Phi(\Psi, \mathcal{G}_t x_t).
\end{aligned}
$$

But, a simple calculation shows that

$$(\Psi, \mathcal{G}_t x_t) + \Psi^T(0)G(t, x_t)$$

and we have the following theorem.

<u>Theorem III.2</u>. Suppose the operators $\mathcal{A}, \mathcal{A}^*$ are defined by (3.4), (3.5), respectively, and let $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$, $\lambda \in P\sigma(\mathcal{A})$, be the maximal subspaces of $C([0, r], R^n)$ annihilated by powers of $(\mathcal{A} - \lambda I)$, $(\mathcal{A}^* - \lambda I)$, respectively. If Φ is a basis for $\mathcal{M}_\lambda(\mathcal{A})$, Ψ is a basis for $\mathcal{M}_\lambda(\mathcal{A}^*)$, (ψ, φ) defined in (3.3), then the change of variables (3.17) applied to system (3.20), (3.21) yields the equivalent system

$$
(3.22)\qquad
\begin{aligned}
\dot{y}(t) &= By(t) + \Psi^T(0)G(t, \Phi y(t) + z_t), \\
\dot{z}_t &= \mathcal{A}z_t + \mathcal{G}_t(\Phi y(t) + z_t) - \Phi\Psi^T(0)G(t, \Phi y(t) + z_t), \\
&(\Psi, z_t) = 0
\end{aligned}
$$

where B is the matrix defined by $\mathcal{A}\Phi = \Phi B$, has all of its eigenvalues equal to λ, and the point spectrum of the operator $\mathcal{A}$ restricted to the set of $\bar{\varphi}$ such that $(\Psi, \bar{\varphi}) = 0$ does not contain λ.

Remark III.1. By a repeated application of the above process, it follows from Corollary III.1 that one make a further decomposition of the space $C([-r, 0], R^n)$ which will yield a system of the form (3.22) where the real parts of the point spectrum of the operator $\mathcal{A}$ restricted to the set of $\bar{\varphi}$ such that $(\Psi, \bar{\varphi}) = 0$ are less than any preassigned value, β. The matrix B will then have its eigenvalues equal to the elements of $P\sigma(\mathcal{A})$ which have a real part $\geq \beta$.

IV. Perturbation of linear systems. In this section, we indicate some applications of the results of the previous section to the system

$$(4.1)\qquad \dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t+\theta) + g(t)$$

where x is an n-vector, $\eta(\theta)$ is an $n \times n$ matrix whose elements are of bounded variation on $[-r, 0]$ and $g(t)$ is continuous on $(-\infty, \infty)$. Together with system (4.1), we consider the adjoint equation

$$(4.2)\qquad \dot{y}(t) = -\int_{-r}^{0} [d\eta^T(\theta)]y(t-\theta).$$

Lemma IV.1. If $\mathcal{T}(t)$, $t \geq 0$, $\mathcal{T}(0) = I$, is a strongly continuous semi-group of operators of a Banach space into itself and if for some $r > 0$, the spectral radius $\rho = \rho_{\mathcal{T}(r)}$ is < 1 and $\neq 0$, then for any function h mapping $(-\infty, \infty)$ into $\mathcal{B}$ such that $h(t)$ is almost periodic, the function

$$(4.3)\qquad z_t^* = \int_{-\infty}^{t} \mathcal{T}(t-\tau)h(\tau)d\tau$$

is almost periodic in t with the same frequencies as $h(t)$, $\|z_t^*\| \leq KR/\beta$, $-2\beta r = \log \rho$, K constant, $R = \sup_t \|h(t)\|$, and is a uniformly asymptotically stable solution of

$$(4.4)\qquad \dot{z}_t = \mathcal{A} z_t + h(t)$$

where $\mathcal{A}$ is the infinitesimal generator of $\mathcal{T}(t)$.

Proof: If $-2\beta r = \log \rho$, and $\gamma = \beta$, then Theorem II.1 implies $\|\mathcal{T}(t-\tau)\varphi\| \leq Ke^{-\beta(t-\tau)}\|\varphi\|$ for all $t \geq \tau$, and some constant K. Therefore, the integral in (4.3) exists and $\|z_t^*\| \leq KR/\beta$. Furthermore,

$\mathcal{J}(0) = I$ and $d\mathcal{J}(t)/dt = \mathcal{Q}\mathcal{J}(t)$ implies z^*_t satisfies (4.4). The uniform asymptotic stability of z^*_t follows from Theorem II.1. To show that z^*_t is almost periodic with the same frequencies as $h(t)$, it is sufficient to show that for every sequence of real numbers $\{\tau_m\}$ for which

$$h(t + \tau_m) - h(t) \to 0 \text{ as } m \to \infty, \text{ uniformly for } t \text{ in } (-\infty, \infty),$$

we have

$$z^*_{t+\tau_m} - z_t \to 0 \text{ as } m \to \infty, \text{ uniformly for } t \text{ in } (-\infty, \infty).$$

This is easily verified using Theorem II.1 and the lemma is proved.

The proof of the next two lemmas are standard and may be found in [7,12].

Lemma IV.2. If C is a constant $n \times n$ matrix whose eigenvalues have real parts $\geq 2\beta > 0$, and $h(t)$ is an almost periodic n-vector, then there is a unique almost periodic solution of the equation

$$\dot{w} = Bw + h(t)$$

which is bounded by KR/β for some constant K and $R = \sup_t |h(t)|$.

Lemma IV.3. If B is a constant $n \times n$ matrix whose eigenvalues may have zero real parts, $h(t)$ is an almost periodic n-vector with a finite Fourier series, then a necessary and sufficient condition that the system

$$\dot{y} = By + h(t)$$

have an almost periodic solution is that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \psi_j(t)h(t)dt = 0, \quad j = 1,2, \ldots, r,$$

where $(\psi_1, \ldots, \psi_r)$ is a basis for the almost periodic solutions of the adjoint equation

$$\dot{y} = - yB$$

One could also state a result similar to Lemma IV.3 for the case when $h(t)$ does not have a finite Fourier series, but it is necessary to have an hypothesis which guarantees that the integrals of certain almost periodic functions are almost periodic. We do not discuss this question here.

Lemmas IV.1 - IV.3 together with Theorem III.1 and Remark III.1 yield in a very natural way extensions to functional-differential equations of the standard results concerning the existence of periodic and almost periodic solutions of ordinary differential equations which are perturbations of a linear system with constant coefficients. We do not state all of these results for functional-differential equations but merely give some indications of the manner in which they are obtained. After inspection of the proofs, one will see that the arguments are essentially the same as for ordinary differential equations.

<u>Theorem 4.1</u>. If $g(t)$ is 2π-periodic and $(\psi_1, \ldots, \psi_k)$ represents a basis for the 2π-periodic solutions of the adjoint equation (4.2), then a necessary and sufficient condition that system (4.1) has a 2π-periodic solution is that

$$\int_0^{2\pi} \psi_k^T(t)g(t)dt = 0.$$

If there are no 2π-periodic solutions of the adjoint equation, then system (4.1) has a unique 2π-periodic solution bounded by KR/β for some constant K, $R = \sup_t |g(t)|$ and 2β being defined as the largest number such that $|\lambda - in| \geq 2\beta$, $n = 0, \pm 1, \pm 2, \ldots,$ for all roots λ of the characteristic equation (2.6).

Proof: Let $\mathcal{A}$, $\mathcal{A}^*$ be the operators defined by (3.4), (3.5). Let Φ_1, Ψ_1 be bases for the linear extensions of the manifolds $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$, respectively, for all λ in $P\sigma(\mathcal{A})$ with $\lambda \not\equiv 0 \pmod{1}$, $\mathrm{Re}\lambda \geq 0$, and let Φ_2, Ψ_2 be bases for the linear extensions of the manifolds $\mathcal{M}_\lambda(\mathcal{A})$, $\mathcal{M}_\lambda(\mathcal{A}^*)$, respectively, for all λ in $P\sigma(\mathcal{A})$ with $\lambda \equiv 0 \pmod{1}$. Furthermore, we can choose $(\Psi_1, \Phi_1) = I$, $(\Psi_2, \Phi_2) = I$, where (ψ, φ) is defined in (3.3) (see Lemma III.7). Consequently, by Theorem III.2 and Remark III.1 the transformation of variables

$$
\begin{aligned}
&x_t = \Phi_1 w(t) + \Phi_2 y(t) + z_t, \\
(4.5)\qquad & \\
&(\Psi_1, x_t) = w(t), \quad (\Psi_2, x_t) = y(t),
\end{aligned}
$$

applied to (4.1) yields the equivalent system

$$
\begin{aligned}
&\dot{w}(t) = B_1 w(t) + \Psi_1^T(0)g(t) \\
(4.6)\qquad &\dot{y}(t) = B_2 y(t) + \Psi_2^T(0)g(t) \\
&\dot{z}(t) = \mathcal{A} z_t + \mathcal{G}_t - \Phi_1\Psi_1^T(0)g(t) - \Phi_2\Psi_2^T(0)g(t), \\
&\qquad\qquad ((\Psi_1, \Psi_2), z_t) = 0,
\end{aligned}
$$

where $\mathcal{G}_t(\theta) = 0$ for $-r \leq \theta < 0$, $= g(t)$ for $\theta = 0$, B_1, B_2 are the matrices defined by $\mathcal{A}\Phi_1 = \Phi_1 B$, $\mathcal{A}\Phi_2 = \Phi_2 B_2$, respectively, all eigenvalues of B_1 are $\not\equiv 0$ (mod i), all eigenvalues of B_2 are $\equiv 0$ (mod i) and all elements in the point spectrum of the restriction of $\mathcal{A}$ to the set of all $\bar{\varphi}$ such that $((\Psi_1, \Psi_2), \bar{\varphi}) = 0$ have negative real parts. System (4.1) will have a 2π-periodic solution if and only if there is a 2π-periodic solution $w(t)$, $y(t)$, z_t of (4.6). After observing that the basis $\psi_1, \ldots, \psi_k$ of the 2π-periodic solutions of the adjoint equation (4.2) are of the form $\Psi\, v_j(t)$ where the $v_j(t)$ are 2π-periodic solutions of the equation $\dot{v} = -B^T v$, one can apply Lemmas IV.1-IV.3 to complete the proof of Theorem IV.1.

In the same manner, one proves

Theorem IV.2. If $g(t)$ is almost periodic in t with a finite Fourier series and $(\psi_1, \ldots, \psi_k)$ represents a basis for the almost periodic solutions of the adjoint equation (4.2), then a necessary and sufficient condition that system (4.1) has an almost periodic solution is that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \psi_j(t)g(t)dt = 0, \quad j = 1,2,\ldots,k.$$

If there is no almost periodic solution of the adjoint equation, then there is a unique almost periodic solution of (4.1) and it is bounded by KR/β for some constant K, $R = \sup_t |g(t)|$, and 2β defined as the largest number such that $|\mathrm{Re}\,\lambda| \geq 2\beta$ for all roots of λ of the characteristic equation (2.6).

No consider the nonlinear equation

$$(4.7) \qquad \dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t+\theta) + G(t, x_t, \epsilon)$$

where $\eta(\theta)$ is the same matrix as before, ϵ is a parameter, $q(t, \varphi, \epsilon)$ is continuous in t, φ, ϵ for $-\infty < t < \infty$, $\varphi \in C([-r, 0], R^n)$, $\|\varphi\| \le H$, $H > 0$, $0 \le \epsilon \le \epsilon_0$, and is Lipschitzian in φ. Furthermore, there exists a function $\eta(\epsilon, \rho)$, continuous for $0 \le \epsilon \le \epsilon_0$, $0 \le \rho \le H$, such that $\eta(0, 0) = 0$ and

$$|G(t, \varphi, \epsilon) - G(t, \psi, \epsilon)| \le \eta(\epsilon, \rho)\|\varphi - \psi\|, \quad q(t, 0, 0) = 0,$$

for all $\varphi, \psi \in C([-r, 0], R^n)$, $\|\varphi\|, \|\psi\| \le \rho$, $0 \le \epsilon \le \epsilon_0$, $-\infty < t < \infty$.

If the characteristic equation (2.6) of $\dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t + \theta)$ has no roots $\equiv 0 \pmod{i}$ and G is 2π-periodic in t, then one can prove that there is a unique 2π-periodic solution of (4.7) in a neighborhood of the origin and this solution approaches zero as $\epsilon \to 0$. If the roots of this characteristic equation have nonzero real parts and $G(t, \varphi, \epsilon)$ is almost periodic in t uniformly with respect to φ for each fixed ϵ, then there is a unique almost periodic solution of (4.7) in a neighborhood of the origin and this solution approaches zero as $\epsilon \to 0$. The proofs may be supplied exactly as in [7; Chapters 5, 12] by making use of the change of variables as in Theorem IV.1. These results have been obtained previously in a slightly less general form (see, for example, Krasovskii [10], Shimanov [16]).

What happens in the case when some of the roots of the characteristic equation (2.6) are $\equiv 0 \pmod{i}$ and G is 2π-periodic in t? In this case, one can extend the method of casting out secular terms to derive the determining equations (bifurcation equations) associated with (4.7).

This procedure has already been indicated by Shimanov [ ], but we repeat it here for completeness. For simplicity in notation, we restrict ourselves to the following case: if

$$\Delta(\lambda) = \lambda I - \int_{-r}^{0} [d\eta(\theta)]e^{\lambda\theta}$$

and λ is the root of multiplicity μ of $\det \Delta(\lambda) = 0$ which is $\equiv 0 \pmod{i}$, then there are μ linearly independent solutions of $\Delta(\lambda)b = 0$. Furthermore, for simplicity only, suppose $G = \epsilon H$.

By the change of variables (4.5), system (4.7) can be written as

$$
\begin{aligned}
\dot{w}(t) &= B_1 w(t) + \epsilon\, \Psi_1^T(0)G_1, \\
\dot{y}(t) &= B_2 y(t) + \epsilon\, \Psi_2^T(0)G_1, \\
\dot{z}_t &= \mathcal{A} z_t + \epsilon \mathcal{G}_{1t} - \epsilon\, \Phi_1\Psi_1^T(0)G_1 - \epsilon\, \Phi_2\Psi_2^T(0)G_1, \\
&((\Psi_1, \Psi_2)z_t) = 0,
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
G_1 &= G_1(t, w(t), y(t), z_t) = \epsilon H(t, \Phi_1 w(t) + \Phi_2 y(t) + z_t), \\
\mathcal{G}_{1t}(\theta) &= \begin{cases} 0, & -r \le \theta < 0, \\ G_1, & \theta = 0, \end{cases}
\end{aligned}
\tag{4.9}
$$

no eigenvalue of B_1 is $\equiv 0 \pmod{i}$, $e^{B_2 t}$ is 2π-periodic and all elements in the point spectrum of the restriction of $\mathcal{A}$ to the set of all $\bar{\varphi}$ such that $((\Psi_1, \Psi_2), \varphi) = 0$ have negative real parts.

If we let $y(t) = e^{B_2 t} v(t)$, system (4.8) can be written as

$$
\begin{aligned}
\dot{w}(t) &= B_1 w(t) + \epsilon \Psi_1^T(0) G_1, \\
\dot{v}(t) &= \epsilon e^{-B_2 t} \Psi_2^T(0) G_1, \qquad (4.10) \\
\dot{z}(t) &= \mathcal{A} z_t + \epsilon \mathcal{G}_{1t} - \epsilon \Phi_1 \Psi_1^T(0) G_1 - \epsilon \Phi_2 \Psi_2^T(0) G_1, \\
&((\Psi_1, \Psi_2), z_t) = 0,
\end{aligned}
$$

where $G_1 = G_1(t, w(t), e^{-B_2 t} v(t), z_t)$. One can now use Lemma IV.1 to repeat all of the arguments in [7, Chapter 6] to obtain a periodic function $(w^*(t), v^*(t), z_t^*)$ with a vector a, the mean value of v^*, arbitrary, and a set of equations (the determining equations) involving the arbitrary vector a which have the property that $(w^*(t), v^*(t), z_t^*)$ will be a periodic solution of (4.10) is and only if the vector a satisfies the determining equations. We do not discuss this question further since all details are easily supplied. This procedure extends the method of Cesari-Hale-Gambill to functional-differential equations of the above type.

In some problems, it is more convenient to introduce polar coordinates for the vector y in (4.8). In the following, we may allow G in (4.2) to be almost periodic in t and need only assume that the matrix B_2 has the form

$$
\begin{aligned}
B_2 &= \operatorname{diag}(A_1, \ldots, A_k) \\
A_j &= \begin{pmatrix} 0 & 1 \\ -\sigma_j^2 & 0 \end{pmatrix}, \quad j = 1, 2, \ldots, k
\end{aligned} \qquad (4.11)
$$

where each σ_j is positive. If $y = (y_1, \ldots, y_{2k})$, the transformation

$$
\begin{aligned}
& y_{2j-1} = \rho_j \sin \theta_j \\
(4.12) \qquad & y_{2j} = \rho_j \cos \theta_j, \quad j = 1,2, \ldots, k, \\
& \rho_1 = (\rho_1, \ldots, \rho_k), \quad \theta = (\theta_1, \ldots, \theta_k)
\end{aligned}
$$

applied to (4.8) yields a set of equations of the form

$$
\begin{aligned}
& \dot\theta(t) = d + \epsilon\, \Theta(t, \theta(t), \rho(t), w(t), z_t, \epsilon) \\
& \dot\rho(t) = \epsilon\, R(t, \theta(t), \rho(t), w(t), z_t, \epsilon) \\
(4.13) \qquad & \dot w(t) = B_1 w(t) + \epsilon\, W(t, \theta(t), \rho(t), w(t), z_t, \epsilon) \\
& \dot z_t = \mathcal{A} z_t + \epsilon\, Z(t, \theta(t), \rho(t), w(t), z_t, \epsilon)
\end{aligned}
$$

where z_t is required to satisfy $((\Psi_1, \Psi_2), z_t) = 0$, and $d = (\sigma_1, \ldots, \sigma_k)$. Equations (4.13) are of the same type that have been considered in [3,7] for ordinary differential equations. It is a simple matter to extend the method of averaging in [3,7] to systems of the form (4.13) and to prove the existence of integral manifolds which are generated by equilibrium points of the averaged equations. We do not go in detail on these general questions, since these extensions will be clear to the reader who is familiar with the results for ordinary differential equations.

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